THE KINEMATICAL METHOD OF TANGENTS.

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1. If a plane B slide in any manner upon a plane A, every point in B describes a locus in the plane A, and every point in A describes a locus in the plane B. It is immaterial which we regard as the fixed and which the moving plane; and the relative motion of the planes is obviously determined when the loci in the other plane of two points, either both in one plane or one in each plane, are given; with the single exception of the case when the loci, one in each plane, are circles of which the given points are the centres. For example, in the "tram motion" two points, P and Q, of the moving plane B describe perpendicular straight lines in A, and this defines a motion in which the middle point O of PQ describes a circle in A whose centre is at O' the intersection of the straight lines; while all points of B situated on the circumference of the circle whose diameter is PQ describe straight lines in A intersecting in the same point O'. Thus the same motion is determined when two points of the plane B describe any two intersecting straight lines, and also when two points describe a circle and one of its diameters. The reciprocal motion of A upon B is that in which two straight lines of the plane now regarded as the moving plane, slide on pins in the fixed planes, the intersection O' of the lines evidently describing a circle passing through the pins; so that the motion may also be defined as that in which a given point O' describes a circle, and a straight line of the moving plane drawn through O' slides upon a fixed point in the circumference of the circle. This is the motion by which the limaçon, or protraction of a circle from a point in its circumference, is described by any point in the moving straight line. Moreover, since the straight line joining any point of the plane A (now the moving plane) to O' passes through a fixed point in the circumference of the circle in B, every point in A describes a limaçon in B, while, as is well known, every point in B describes an ellipse in A.

2. More generally, whatever be the character of the motion, any curve in the moving plane B has an envelope in the plane A, which it touches in one or more points. Confining our attention to a single point of contact, we may consider the condition that a given curve in B shall touch a given curve in A, and it is evident that in general two such conditions determine the relative motion of A and B. When one of the pair of touching curves reduces to a point, the condition becomes that considered above in which a given point in one plane describes a given curve in the other.

When two curves touch one another, a curve parallel to one of them and at a distance c from it necessarily touches a parallel to the other curve at a distance c from it; thus the condition that the parallel shall form a touching pair is equivalent to the condition that the original curves shall touch. For example, the

condition that a circle in B shall touch a given curve in A is equivalent to the condition that a concentric circle in B shall touch one of the parallels to the curve in A, and in particular to the condition that the centre of the circle in B shall describe a certain one of these parallels in the plane A. Two equivalent conditions of course fail to determine the relative motion of the planes, as in the exceptional case considered in Art. 1.

3. The simplest kind of relative motion is that of simple rotation in which both the curves of a touching pair reduce to points; in this case each of the equivalent conditions consists in the coincidence of equal circles, sliding one upon the other, and forming the "turning pair" of the Kinematics of Machinery. This is the only case in which a single pairing determines the motion; but, as a special case, we have that in which the centre is at an infinite distance, and the sliding circles become straight lines forming the "sliding pair."

4. There is at every instant of the motion an instantaneous centre of rotation; that is to say, either plane being regarded as fixed, a point in it about which the other plane is at the instant rotating. The point of the moving plane which is at the instantaneous centre has of course no motion at the instant, but, except in the case of simple rotation considered above, the instantaneous centre shifts its position continually in each plane.

5. If from the instantaneous centre we draw a normal to any curve in the moving plane B, it is evident that the foot of the normal is the point of contact of the curve in B with its envelope in A. Thus the common normal at the point of contact of a pair of touching curves passes through the instantaneous centre; and, if in a given relative position of the planes we draw the normals corresponding to two pairs of touching curves, we shall generally determine the instantaneous centre. If, however, the touching pairs are the equivalent pairs of Art. 2, the two normals are coincident and fail to determine the point. If one of a touching pair reduces to a point, as is frequently the case with the pairs used to define the motion, the normal is determined by the locus of the point. Having determined the instantaneous centre, the normal drawn to any curve in the moving plane B determines its point of contact with its envelope in the plane A, and is also normal to that envelope; and, in particular, the line joining the instantaneous centre to any point of B is the normal to the locus of this point in the fixed plane A.

6. Tangents to a considerable number of ordinary curves may thus be determined. For example, the process when applied to the ellipse regarded as generated by the "tram motion" described in Art. 1 is as follows: Inscribe between the given point and the major axis a line equal to the minor semi-axis, produce the line to meet the minor axis, draw perpendiculars to the axes from the points thus determined; then the intersection of these perpendiculars is a

point on the normal to the ellipse at the given point. For the conchoid the process is as follows: Draw a radius vector through the given point to meet the directrix; the intersection of a perpendicular to the directrix at this point with a perpendicular to the radius vector at the pole determines the instantaneous centre and the normal at the given point. A similar construction applies to all curves defined as protractions of a given curve. In the case of the limaçon and cardioid the instantaneous centre is the point of the circular directrix opposite to that in which it is cut by the radius vector.

Again, in the three-bar motion, the instantaneous centre for the motion of the middle bar is the intersection of the other two bars. This point determines the normal to the locus of every point on or rigidly connected with the middle bar. In the motion of a crank and connecting-rod, the instantaneous centre is the intersection of the crank with a perpendicular through the piston end of the connecting-rod to the line of its motion, and this point determines the normals to the ovals described by points in the connecting-rod.

7. The loci of the instantaneous centre in the two planes are called the *centroids* of the relative motion. Since the points of the two planes which at any instant coincide with the instantaneous centre are relatively at rest, it is evident that the instantaneous centre moves at the same rate in each plane; thus the centroids roll one upon the other, the point of contact at any instant being the instantaneous centre.

8. A single pair of touching curves together with the law governing the rates at which the point of contact travels on each curve would determine the relative motion of the planes. Two particular cases may be noticed. First, that in which the point of contact travels at the same rate in each curve, in which case the curves are the centroids. Second, that in which the rate is zero in one curve; thus let a curve in the plane B remain always in contact with a given curve in A, the point of contact being a fixed point of B. The common normal is now a fixed line of B, and its envelope in A is the evolute of the given curve. Thus the fixed line in B and this evolute are the centroids of the motion, and the instantaneous centre is the centre of curvature of the given curve in A. For example, the tractrix is described by one extremity of a straight line to which it is always tangent, while the other extremity describes the directrix, and a perpendicular to the directrix at this point meets the corresponding normal to the tractrix in the centre of curvature.

[The consideration of the instantaneous centre and the beginning of kinematical methods in Geometry date from Descartes who applied the process in his study of the cycloid (*Lettres* (Aug. 23, 1638) VII, 89. Ed. Cousin). The discovery of the existence of an instantaneous centre for any motion of a plane system is due to John Bernoulli (*Opera* 4, 265). Poinsot introduced the use of cen-

troids and axoids (*Liouville's Journal*, XVI, 9–129; 286–336). The application of these methods to the study of Mechanisms was first made systematically by Willis (1841); his processes were improved by Rankine (*Machinery and Millwork*, 1871); and the whole subject was reformed and freshly stated by Reuleaux (*Kinematik*, 1874), and has since been still further developed by Grashof and others. Peculiar interest attaches to the "tram-motion" from its furnishing to Leonardo da Vinci his famous discovery of the elliptic chuck for turning ovals on the lathe (Chasles, *Aperçu* 531).—*W. M. T.*]

A COLLECTION OF FORMULÆ FOR THE AREA OF A PLANE TRIANGLE.*

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In April, 1883, Mr. James Main, formerly of the U. S. Coast and Geodetic Survey, published in the *Mathematical Magazine* a collection of forty-six (46) expressions for the area of a plane triangle, prefacing it with the remark that this collection "may be regarded as a matter of curiosity," and that about one-half of the formulæ are well known.

In the following August M. Ed. Lucas reprinted this collection in *Mathesis* in a classified form, separating the formulæ into five groups and adding one formula not contained in Mr. Main's list. The collection has also been reprinted in the third number of the *Tidsskrift for Mathematik*, 1883. Some two or three additional formulæ have since been printed in various mathematical publications.

The terms in which the area is expressed in Mr. Main's collection are angles, sides, perpendiculars, and radii of inscribed, escribed, and circumscribed circles. No formulæ are given involving medians or bisectors. In numbering Mr. Main has not counted those formulæ as distinct which arise from merely permuting the letters, nor has he in *every* case given all the forms possible to be obtained by permuting the letters, though he has generally done so. As numbered, then, he counts forty-six formulæ, but if every form be counted as a distinct one the total number is ninety-four.

M. Lucas, by making all possible permutations and adding one new form, makes the number 139, to which some two or three have been added since.

As the matter has proved of interest, the following collection has been made, which is a still further extension; the additional formulæ being chiefly due to introducing the medians and bisectors, not used in the former collections. In this collection Mr. Main's mode of numbering has been followed and formulæ

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